

A SUFFICIENT CONDITION FOR INTRINSIC KNOTTING OF BIPARTITE GRAPHS

SOPHY HUCK, ALEXANDRA APPEL, MIGUEL-ANGEL MANRIQUE, AND THOMAS W. MATTMAN

Dedicated to the memory of Michael Curtis Wilson.

ABSTRACT. We present evidence in support of a conjecture that a bipartite graph with at least five vertices in each part and $|E(G)| \geq 4|V(G)| - 17$ is intrinsically knotted. We prove the conjecture for graphs that have exactly five or exactly six vertices in one part. We also show that there is a constant C_n such that a bipartite graph with exactly $n \geq 5$ vertices in one part and $|E(G)| \geq 4|V(G)| + C_n$ is intrinsically knotted. Finally, we classify bipartite graphs with ten or fewer vertices with respect to intrinsic knotting.

1. INTRODUCTION

We recently discovered [CMOPRW] that a result of Mader [M] leads to a proof of a conjecture of Sachs: A graph on $|V(G)| \geq 7$ vertices with at least $5|V(G)| - 14$ edges is intrinsically knotted. In the current paper, we will give evidence in support of a similar bound $|E(G)| \geq 4|V(G)| - 17$ that ensures knotting of bipartite graphs.

Recall that a graph is intrinsically knotted (IK) if every tame embedding of the graph in \mathbb{R}^3 contains a non-trivially knotted cycle. Since knotless embedding is preserved under edge contraction [NT], work of Robertson and Seymour [RS] shows that this property is determined by a finite list of minor minimal IK graphs. However, determining this list remains difficult. It is known [CG, F1, KS, MRS] that K_7 and $K_{3,3,1,1}$ along with any graph obtained from these two by triangle-Y exchanges is minor minimal with respect to intrinsic knotting. Foisy [F2, F3] has shown the existence of several additional minor minimal IK graphs.

As a complete characterization of IK graphs remains out of reach for the moment, we propose instead a sufficient condition for intrinsic knotting that is easy to test. For a graph G , let $V(G)$ denote the set of vertices and $v(G) = |V(G)|$ the number of vertices. Similarly, $e(G) = |E(G)|$ will be the number of edges.

Conjecture 1. *Let G be a bipartite graph with at least five vertices in each part. If $e(G) \geq 4v(G) - 17$ then G is intrinsically knotted.*

Note that if a bipartite graph has four or fewer vertices in one part, then it is not IK [BBFFHL]. Also, any graph can be made bipartite without affecting its topology by adding degree two vertices in the middle of selected edges. So, even for graphs that are not bipartite, the proposed bound may be more useful in some

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instances than the bound $e(G) \geq 5v(G) - 14$ that applies to all graphs of seven or more vertices.

The conjecture was proved for graphs with exactly five vertices in one part by [CHPS]. We build on their work to show

Theorem 2. *Let G be a bipartite graph with exactly five or exactly six vertices in one part and at least five vertices in the other. If $e(G) \geq 4v(G) - 17$ then G is IK. Moreover, this is also true if G has exactly seven vertices in each of its two parts.*

For bipartite graphs of ten vertices, the bound $e(G) \geq 4v(G) - 17 = 23$ almost characterizes intrinsic knotting. Let $K_{n_1, n_2} \setminus m$ denote the set of graphs constructed from the complete bipartite graph K_{n_1, n_2} by removing m edges. We will use a_1, a_2, \dots, a_{n_1} to denote the vertices in one part (the *a-vertices*) and b_1, b_2, \dots, b_{n_2} those in the other (the *b-vertices*).

Theorem 3. *A bipartite graph on ten or fewer vertices is not IK unless it has five vertices in each part. A graph G of the form $K_{5,5} \setminus m$ is IK if and only if*

- G has 23 or more edges, or
- G is the graph $K_{5,5}$ with the edges a_1b_1 , a_1b_2 , and a_2b_1 removed.

We will use the notation $K_{5,5} \setminus \{a_1b_1, a_1b_2, a_2b_1\}$ to describe the unique IK $K_{5,5} \setminus 3$ graph.

Theorem 3 will be proved in Section 3 below, along with theorems that give sufficient conditions for intrinsic knotting of subgraphs of $K_{6,6}$ and $K_{7,7}$. In Section 4, we use these results to prove Theorem 2 as well as theorems that, together, provide sufficient conditions for intrinsic knotting of subgraphs of any complete bipartite graph.

Although we are unable to prove our conjecture, in Section 5 we will show that a bound of the form $e(G) \geq 4v(G) + C_n$ ensures intrinsic knotting for bipartite graphs with exactly $n \geq 5$ vertices in one part:

Theorem 4. *Let a_n be defined by the recurrence*

$$a_n = \left\lfloor \frac{n(a_{n-1} - 1)}{n - 5} \right\rfloor + 1$$

when $n \geq 7$, $a_5 = 5$, and $a_6 = 7$. Let $C_n = a_n - 4n$, for $n \geq 7$, and $C_5 = C_6 = -17$. Let G be a bipartite graph with exactly $n \geq 5$ vertices in one part and at least a_n vertices in the other. If $e(G) \geq 4v(G) + C_n$ then G is IK.

We remark that our techniques for showing that a graph is IK ultimately come down to showing that it has one of the graphs obtained from K_7 by triangle-Y moves as a minor. In particular, we will make use of the graphs F_9 and H_9 constructed in this way by Kohara and Suzuki [KS]. This means we have no new examples of minor minimal IK graphs. In particular, we deduce that there are no new examples to be found among bipartite graphs on ten or fewer vertices.

This paper is largely an abbreviated version of [HAM] to which we refer the reader for additional details.

2. LEMMAS

Our analysis of intrinsic knotting of bipartite graphs is based primarily on the following lemma which follows easily from the Pigeonhole Principle.

Lemma 5. *Let a, b, k , and m be positive integers such that $(k-1)(a+1) < m+k$. Every graph of the form $K_{a+1,b} \setminus (m+k)$ has a subgraph of the form $K_{a,b} \setminus m$.*

Recall that any graph obtained from the graph G by a sequence of edge deletions or contractions is called a minor of G . It follows from [NT] that if H is IK and H is a minor of G , then G is also IK. Since subgraphs are examples of minors, Lemma 5 implies

Lemma 6. *Let a, b, k , and m be positive integers such that $(k-1)(a+1) < m+k$. If every graph of the form $K_{a,b} \setminus m$ is IK, then the same is true of every graph of the form $K_{a+1,b} \setminus (m+k)$.*

We also make note of a useful lemma due to [BBFFHL, OT]. Let $K_2 + G$ denote the join of G and K_2 , the complete graph on 2 vertices.

Lemma 7. *$K_2 + G$ is IK if and only if G is non-planar.*

3. SUBGRAPHS OF $K_{5,5}$, $K_{6,6}$, AND $K_{7,7}$

In this section we prove Theorem 3, characterizing intrinsic knotting of bipartite graphs on 10 or fewer vertices, as well as theorems that give sufficient conditions for a subgraph of $K_{6,6}$ or $K_{7,7}$ to be IK.

Theorem 3. *A bipartite graph on ten or fewer vertices is not IK unless it has five vertices in each part. A graph G of the form $K_{5,5} \setminus m$ is IK if and only if*

- G has 23 or more edges, or
- $G = K_{5,5} \setminus \{a_1b_1, a_1b_2, a_2b_1\}$.

Proof: It was shown in [BBFFHL] that a bipartite graph with 4 or fewer vertices in one part is not intrinsically knotted. In [CMOPRW] it is argued that all graphs of the form $K_{5,5} \setminus 2$ are IK, so any graph $K_{5,5} \setminus m$ with 23 or more edges is IK.

There are four graphs of the form $K_{5,5} \setminus 3$. Three of these are subgraphs of a graph of the form $K_2 + H$ where H is a planar graph and are not IK by Lemma 7. The fourth, $K_{5,5} \setminus \{a_1b_1, a_1b_2, a_2b_1\}$ was shown in Figure 8 of [MOR] to have the IK graph H_9 as a minor and is therefore also intrinsically knotted. Here, H_9 is one of the graphs obtained by triangle-Y substitution on K_7 ; see [KS].

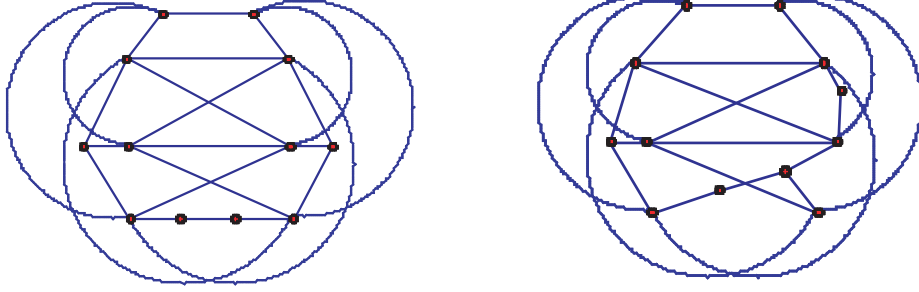
All but one of the $K_{5,5} \setminus 4$ graphs are subgraphs of an unknotted $K_{5,5} \setminus 3$ and are, therefore, also not IK. The remaining graph, $K_{5,5} \setminus \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$ is, again, a subgraph of a $K_2 + H$ graph with H planar, so not IK. Since no $K_{5,5} \setminus 4$ graph is IK, no $K_{5,5} \setminus m$ graph with $m \geq 4$ is IK. \square

The following two theorems make use of graphs F_{66} and H_{66} (see Figure 1) of the form $K_{6,6} \setminus 12$ that have as minors the graphs F_9 and H_9 obtained from K_7 by triangle-Y moves in [KS]. Since F_9 and H_9 are IK, their expansions F_{66} and H_{66} are also.

Theorem 8. *Every graph of the form $K_{6,6} \setminus 5$ is IK.*

Proof: We will list all ways to remove five edges from the complete bipartite graph $K_{6,6}$ and demonstrate that every case but one (the first case below) has H_{66} as a subgraph. However, the one exceptional case instead has F_{66} as a minor. Thus, in any case, a $K_{6,6} \setminus 5$ graph has an IK minor and is, therefore, IK.

To determine all possible ways to remove five edges from $K_{6,6}$, let a_1, \dots, a_6 be the vertices in one part and b_1, \dots, b_6 those in the other part. Now consider any

FIGURE 1. The $K_{6,6} \setminus 12$ graphs H_{66} (at left) and F_{66} (at right).

partition of 5 and take the i -th element in a partition to be the number of edges removed from the i -th vertex in the first part of $K_{6,6}$. Likewise, consider another partition and allow its entries to correspond to the number of edges removed from the other part of $K_{6,6}$. Some combinations cannot be realized as $K_{6,6} \setminus 5$ graphs (for instance, the pairing $\{\{5\}, \{2, 2, 1\}\}$ cannot be constructed). Observe that there may be more than one graph for a given pairing. Below we indicate the pair of partitions of 5 for each of the twenty $K_{6,6} \setminus 5$ graphs.

Many of these graphs can also be shown to be IK using Corollary 2.5 of [CHPS] which states that a $K_{6,5}$ graph with two or fewer edges removed is IK. In those cases, the list below specifies which vertex to remove to arrive at such a subgraph of $K_{6,5}$.

- (1) $K_{6,6} \setminus \{a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5\}$; $\{\{1, 1, 1, 1, 1\}, \{1, 1, 1, 1, 1\}\}$
- (2) $K_{6,6} \setminus \{a_1b_5, a_2b_5, a_4b_4, a_5b_1, a_6b_6\}$; $\{\{2, 1, 1, 1\}, \{1, 1, 1, 1, 1\}\}$
- (3) $K_{6,6} \setminus \{a_1b_5, a_2b_5, a_4b_1, a_5b_1, a_6b_6\}$; $\{\{2, 2, 1\}, \{1, 1, 1, 1, 1\}\}$
- (4) $K_{6,6} \setminus \{a_1b_5, a_2b_5, a_5b_1, a_5b_2, a_6b_6\}$; $\{\{2, 1, 1, 1\}, \{1, 1, 2, 1\}\}$
- (5) $K_{6,6} \setminus \{a_2b_5, a_4b_1, a_5b_1, a_5b_2, a_6b_6\}$; $\{\{2, 1, 1, 1\}, \{1, 2, 1, 1\}\}$
- (6) $K_{6,6} \setminus \{a_1b_5, a_2b_5, a_5b_1, a_5b_2, a_6b_6\}$; $\{\{2, 1, 2\}, \{1, 2, 1, 1\}\}$
- (7) $K_{6,6} \setminus \{a_1b_5, a_2b_5, a_5b_1, a_5b_2, a_5b_3\}$; $\{\{2, 1, 1, 1\}, \{1, 1, 3\}\}$ Remove vertex a_3 to obtain a $K_{6,5} \setminus 2$ graph which we know to be IK [CHPS, Corollary 2.5].
- (8) $K_{6,6} \setminus \{a_1b_4, a_1b_5, a_5b_1, a_5b_2, a_5b_3\}$; $\{\{2, 3\}, \{1, 1, 1, 1, 1\}\}$ Remove vertex a_3 [CHPS].
- (9) $K_{6,6} \setminus \{a_1b_5, a_4b_1, a_4b_4, a_4b_5, a_5b_1\}$; $\{\{3, 1, 1\}, \{1, 2, 2\}\}$ Remove vertex a_1 [CHPS].
- (10) $K_{6,6} \setminus \{a_4b_1, a_4b_4, a_4b_5, a_5b_1, a_6b_6\}$; $\{\{3, 1, 1\}, \{1, 1, 2, 1\}\}$ Remove vertex a_1 [CHPS].
- (11) $K_{6,6} \setminus \{a_4b_4, a_5b_1, a_5b_2, a_5b_3, a_6b_6\}$; $\{\{3, 1, 1\}, \{1, 1, 1, 1, 1\}\}$ Remove vertex a_1 [CHPS].
- (12) $K_{6,6} \setminus \{a_1b_4, a_1b_5, a_5b_1, a_5b_2, a_5b_4\}$; $\{\{2, 3\}, \{1, 2, 1, 1\}\}$ Remove vertex a_3 [CHPS].
- (13) $K_{6,6} \setminus \{a_1b_4, a_5b_1, a_5b_2, a_5b_3, a_5b_4\}$; $\{\{1, 4\}, \{2, 1, 1, 1\}\}$ Remove vertex a_2 [CHPS].
- (14) $K_{6,6} \setminus \{a_5b_1, a_5b_2, a_5b_3, a_5b_4, a_6b_6\}$; $\{\{4, 1\}, \{1, 1, 1, 1, 1\}\}$ Remove vertex a_1 [CHPS].

- (15) $K_{6,6} \setminus \{a_1b_1, a_1b_2, a_1b_3, a_1b_4, a_1b_5\} ; \{\{5\}, \{1, 1, 1, 1, 1\}\}$ Remove the vertex a_1 from the original graph to obtain a complete $K_{6,5}$ graph. This graph has a $K_{5,5}$ minor, which is known to be IK [S].
- (16) $K_{6,6} \setminus \{a_1b_4, a_4b_1, a_4b_5, a_5b_1, a_5b_4\} ; \{\{1, 2, 2\}, \{2, 2, 1\}\}$
- (17) $K_{6,6} \setminus \{a_1b_4, a_4b_1, a_5b_1, a_5b_4, a_6b_6\} ; \{\{1, 2, 1, 1\}, \{2, 2, 1\}\}$
- (18) $K_{6,6} \setminus \{a_1b_4, a_1b_5, a_2b_5, a_4b_4, a_4b_5\} ; \{\{2, 2, 1\}, \{2, 3\}\}$ Remove vertex b_2 [CHPS].
- (19) $K_{6,6} \setminus \{a_1b_4, a_1b_5, a_4b_4, a_4b_5, a_6b_6\} ; \{\{2, 2, 1\}, \{2, 2, 1\}\}$
- (20) $K_{6,6} \setminus \{a_1b_4, a_4b_1, a_4b_4, a_4b_5, a_5b_4\} ; \{\{1, 3, 1\}, \{1, 3, 1\}\}$ Remove vertex a_2 [CHPS].

□

Theorem 9. *A graph of the form $K_{6,6} \setminus 6$ is IK provided it is not the graph $G_{666} = K_{6,6} \setminus \{a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6\}$.*

Remark: We do not know whether G_{666} is IK or not. Using a computer algebra system, we have verified that none of the graphs obtained by triangle-Y exchanges on K_7 or $K_{3,3,1,1}$ is a minor of G_{666} . So, if G_{666} is IK, it or one of its minors would be a new example of a minor minimal IK graph.

Proof: Let G be a graph of the form $K_{6,6} \setminus 6$. If G has a vertex from which three edges have been removed (compared to the complete bipartite graph $K_{6,6}$), we may delete that vertex to obtain an IK subgraph of the form $K_{6,5} \setminus m$ with $m \leq 3$ [CHPS]. So, we may assume that each vertex in G has at most two edges removed. Thus, it will be enough to consider partitions of six into parts of size one or two. Below we list the seventeen graphs of this form.

- (1) $K_{6,6} \setminus \{a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6\} ; \{\{1, 1, 1, 1, 1, 1\}, \{1, 1, 1, 1, 1, 1\}\}$
- (2) $K_{6,6} \setminus \{a_1b_4, a_2b_5, a_4b_1, a_4b_4, a_5b_2, a_6b_6\} ; \{\{2, 1, 1, 1, 1\}, \{1, 1, 1, 1, 1, 1\}\}$
- (3) $K_{6,6} \setminus \{a_1b_4, a_2b_5, a_3b_5, a_4b_1, a_5b_4, a_6b_6\} ; \{\{2, 2, 1, 1\}, \{1, 1, 1, 1, 1, 1\}\}$
- (4) $K_{6,6} \setminus \{a_1b_5, a_2b_5, a_4b_1, a_4b_4, a_5b_3, a_6b_6\} ; \{\{2, 1, 1, 1, 1\}, \{1, 1, 2, 1, 1\}\}$
- (5) $K_{6,6} \setminus \{a_1b_4, a_2b_5, a_4b_1, a_4b_4, a_5b_2, a_6b_6\} ; \{\{2, 1, 1, 1, 1\}, \{1, 2, 1, 1, 1\}\}$
- (6) $K_{6,6} \setminus \{a_1b_4, a_3b_5, a_4b_1, a_5b_1, a_5b_4, a_6b_6\} ; \{\{2, 2, 1, 1\}, \{1, 2, 1, 1, 1\}\}$
- (7) $K_{6,6} \setminus \{a_1b_4, a_2b_5, a_3b_5, a_5b_1, a_5b_4, a_6b_6\} ; \{\{2, 1, 2, 1\}, \{1, 2, 1, 1, 1\}\}$
- (8) $K_{6,6} \setminus \{a_1b_4, a_2b_5, a_3b_5, a_4b_1, a_5b_1, a_5b_4\} ; \{\{2, 2, 2\}, \{1, 2, 1, 1, 1\}\}$
- (9) $K_{6,6} \setminus \{a_2b_5, a_3b_5, a_4b_1, a_4b_4, a_5b_1, a_5b_2\} ; \{\{2, 2, 1, 1\}, \{1, 2, 1, 2\}\}$
- (10) $K_{6,6} \setminus \{a_3b_1, a_3b_5, a_5b_2, a_5b_4, a_6b_3, a_6b_6\} ; \{\{2, 2, 2\}, \{1, 1, 1, 1, 1, 1\}\}$
- (11) $K_{6,6} \setminus \{a_1b_4, a_2b_5, a_3b_5, a_4b_4, a_5b_1, a_5b_2\} ; \{\{2, 1, 1, 2\}, \{1, 1, 2, 1, 1\}\}$
- (12) $K_{6,6} \setminus \{a_1b_4, a_2b_5, a_4b_1, a_4b_5, a_5b_2, a_5b_4\} ; \{\{2, 1, 2, 1\}, \{1, 2, 1, 2\}\}$
- (13) $K_{6,6} \setminus \{a_1b_4, a_1b_5, a_2b_5, a_5b_1, a_5b_4, a_6b_6\} ; \{\{2, 2, 1, 1\}, \{1, 2, 2, 1\}\}$
- (14) $K_{6,6} \setminus \{a_1b_4, a_2b_5, a_4b_1, a_4b_5, a_5b_1, a_5b_4\} ; \{\{2, 2, 2\}, \{1, 2, 2, 1\}\}$
- (15) $K_{6,6} \setminus \{a_1b_5, a_1b_4, a_4b_4, a_4b_5, a_5b_1, a_6b_6\} ; \{\{2, 2, 1, 1\}, \{2, 2, 1, 1\}\}$
- (16) $K_{6,6} \setminus \{a_1b_5, a_2b_5, a_4b_1, a_4b_4, a_5b_1, a_5b_4\} ; \{\{2, 2, 2\}, \{2, 2, 1, 1\}\}$
- (17) $K_{6,6} \setminus \{a_1b_4, a_1b_5, a_4b_1, a_4b_5, a_5b_1, a_5b_4\} ; \{\{2, 2, 2\}, \{2, 2, 2\}\}$

All but two of the graphs have H_{66} as a subgraph. The two exceptions are the eleventh graph $K_{6,6} \setminus \{a_1b_4, a_2b_5, a_3b_5, a_4b_4, a_5b_1, a_5b_2\}$, which instead has F_{66} as a subgraph, and the first graph G_{666} . □

Theorem 10. *Any graph of the form $K_{7,7} \setminus 10$ is IK.*

Proof: Let G be a graph of the form $K_{7,7} \setminus 10$. If G has a vertex from which three edges have been removed, that vertex may be deleted to obtain a $K_{7,6} \setminus m$ subgraph

with $m \leq 7$. We will show that such graphs are IK in Theorem 11, below. So, we may assume that each vertex of G has at most two edges removed.

Suppose there are vertices a and b , one in each part, each having two edges removed, and such that ab is an edge of G . We will argue that G is IK in this case. This will complete the argument as there must be two vertices with these properties. Indeed, as there are 10 edges removed and G has 7 vertices on each side, there must be at least three vertices on each side with two edges removed. Call them a_1, a_2, a_3, b_1, b_2 , and b_3 . If there are no pair a_i, b_j such that $a_i b_j$ is in G , then each of these six vertices in fact has three edges removed, violating an earlier assumption.

So, we may assume that there are vertices a_1 and b_1 , each with two edges removed and such that $a_1 b_1$ is an edge of G . Then removing a_1 and b_1 from G results in a $K_{6,6} \setminus 6$ graph. If that graph is not G_{666} , then, by Theorem 11, G has an IK subgraph and is itself IK.

Thus, we may assume that removing a_1 and b_1 results in G_{666} . Label the remaining vertices of G so that the edges $a_2 b_2, a_3 b_3, \dots, a_7 b_7$ are not in G . Let $a_1 b_2$ and $a_1 b_3$ denote the edges removed from a_1 . That is, $a_1 b_2$ and $a_1 b_3$ are not edges of G . Then, up to relabeling, there are three possibilities for the edges removed from b_1 . In each case we will show how removing a different pair of vertices from G will result in a $K_{6,6} \setminus 6$ subgraph other than G_{666} .

The first case is that $a_2 b_1$ and $a_3 b_1$ are the edges removed from b_1 . Then, we can obtain a $K_{6,6} \setminus 6$ graph different from G_{666} by removing the vertices a_4 and b_2 instead of a_1 and b_1 . That means, in this case, G has an IK subgraph and is also IK.

Next, suppose that $a_2 b_1$ and $a_4 b_1$ are the edges removed from b_1 . Here, we can obtain a $K_{6,6} \setminus 6$ graph other than G_{666} by removing instead the vertices a_1 and b_3 .

Finally, if $a_4 b_1$ and $a_5 b_1$ are the edges removed from b_1 , then we should instead remove the vertices a_4 and b_2 . \square

4. SUFFICIENT CONDITIONS FOR SUBGRAPHS OF ANY COMPLETE BIPARTITE GRAPH

In this section we present some general results that, taken together with those of the previous section, provide sufficient conditions for intrinsic knotting of a subgraph of $K_{a,b}$ for any $a, b \geq 6$. Together with [CHPS] (who treat the case where one of a or b is exactly five) and [BBFFHL] (who show that a bipartite graph with at most four vertices in one part is not IK) this gives a collection of sufficient conditions for subgraphs of any complete bipartite graph. We will also prove Theorem 2 which shows that our conjecture holds for subgraphs of $K_{7,7}$ as well as bipartite graphs having exactly five or six vertices in one part.

Theorem 11. *All graphs of the form $K_{6+n,6} \setminus (2n+5)$ with $n \geq 1$ are IK.*

Proof: We use induction on n . For the base step, let $n = 1$. We will look at four cases. In each case we will see that there is a way to remove a vertex to obtain an IK $K_{6,6} \setminus m$ subgraph.

Case 1: Consider the $K_{7,6} \setminus 7$ graph where each of the seven a -vertices has exactly one edge removed, and exactly one of the six b -vertices (call it b_1) has exactly 2 edges removed.

We can remove one vertex to obtain a $K_{6,6} \setminus 6$ subgraph, all of which have been shown (Theorem 9) to be IK, except G_{666} . We can avoid G_{666} by not removing one

of the two a -vertices that is connected to b_1 in the complement graph. Removing any of the other a -vertices will leave us with a $K_{6,6} \setminus 6$ graph that is IK.

Case 2: Consider the $K_{7,6} \setminus 7$ graphs where each of the seven a -vertices has exactly one edge removed and at least one of the b -vertices has three or more edges removed. Removing any a -vertex results in a $K_{6,6} \setminus 6$ subgraph that is known to be IK by Theorem 9.

Case 3: Consider the $K_{7,6} \setminus 7$ graphs where each of the seven a -vertices has exactly one edge removed and at least two b -vertices have at least two missing edges. Removing any a -vertex results in a $K_{6,6} \setminus 6$ subgraph that is known to be IK.

Case 4: Consider the $K_{7,6} \setminus 7$ graphs where one or more of the seven a -vertices has more than one edge removed. By removing one of those vertices, we will be left with a $K_{6,6} \setminus m$ subgraph where $m \leq 5$, all of which are known to be IK by Theorem 8.

As these four cases cover all possibilities, all $K_{7,6} \setminus 7$ graphs are IK.

Now, for the inductive step, let $n \geq 1$ and assume all $K_{6+n,6} \setminus (2n+5)$ graphs are IK. Using Lemma 6 we see that every $K_{6+n+1,6} \setminus (2(n+1)+5)$ graph G is also IK.

Thus, for $n \geq 1$, every $K_{6+n,6} \setminus (2n+5)$ graph is IK. \square

Theorem 8 shows that the above result is also true when $n = 0$.

Corollary 12. *Let G be a bipartite graph G with exactly six vertices in one part and at least six vertices in the other part. If $e(G) \geq 4v(G) - 17$, then G is IK.*

We can now prove Theorem 2.

Theorem 2. *Let G be a bipartite graph with exactly five or exactly six vertices in one part and at least five vertices in the other. If $e(G) \geq 4v(G) - 17$ then G is IK. Moreover, this is also true if G has exactly seven vertices in each of its two parts.*

Proof: The case of exactly five vertices was proved in [CHPS]. The case of exactly six vertices is the corollary above while the subgraph of $K_{7,7}$ case is Theorem 10. \square

Theorem 13. *All graphs of the form $K_{7+n,7} \setminus (2n+10)$ with $n \geq 1$ are IK.*

Proof: We use induction on n .

For the base case, let $n = 1$. This gives us a $K_{8,7} \setminus 12$ graph. From Lemma 6 and Theorem 10 we know that all $K_{8,7} \setminus 12$ graphs are IK.

Now, for the inductive step, let $n \geq 1$ and assume all $K_{7+n,7} \setminus (2n+10)$ graphs are IK. By Lemma 6, every $K_{7+n+1,7} \setminus (2(n+1)+10)$ graph is also IK. \square

Note that Theorem 10 corresponds to the case $n = 0$ of this theorem.

Although we cannot verify our conjecture in this case, Theorem 13 does allow us to improve on the previous best known bound (for graphs in general) [CMOPRW] of $e(G) \geq 5v(G) - 14$:

Corollary 14. *Let G be a bipartite graph with exactly seven vertices in one part and at least seven vertices in the other part. If $e(G) \geq 5v(G) - 31$ then G is IK.*

Theorem 15. *All graphs of the form $K_{8+n,8} \setminus (2n+15)$ with $n \geq 1$ are IK.*

Proof: We use induction on n .

For the base case, let $n = 1$. This gives us a $K_{9,8} \setminus 17$ graph G . Applying Lemma 6, we obtain a $K_{9,7} \setminus 14$ subgraph, which is IK by Theorem 13. Therefore G is IK.

Now, for the inductive step, let $n \geq 1$ and assume all $K_{8+n,8} \setminus (2n+15)$ graphs are IK. From Lemma 6, we see that every $K_{8+n+1,8} \setminus (2(n+1)+15)$ graph is also IK. \square

Although we cannot prove the analogue of Theorem 15 when $n = 0$, it follows from Theorem 13 with $n = 1$ and Lemma 6 with $k = 2$ that all $K_{8,8} \setminus 14$ graphs are IK.

Theorem 16. *Any graph of the form $K_{a,a} \setminus (6a - 34)$, where $a \geq 9$, is IK.*

Proof: We use induction on a .

For the base case, let $a = 9$. Consider a $K_{9,9} \setminus 20$ graph. By Theorem 15, every $K_{9,8} \setminus 17$ graph is IK. Then, by Lemma 6, every $K_{9,9} \setminus 20$ is also IK.

Now, for the inductive step, let $a \geq 9$ and assume all $K_{a,a} \setminus (6a - 34)$ graphs are IK. We will show that every $K_{a+1,a+1} \setminus (6(a+1) - 34)$ graph is IK.

Applying Lemma 6 shows that every $K_{a+1,a} \setminus (6a - 31)$ graph is IK. Another application shows that all $K_{a+1,a+1} \setminus (6a - 28)$ graphs are IK. Since $6a - 28 = 6(a+1) - 34$, every $K_{a+1,a+1} \setminus (6(a+1) - 34)$ graph is IK.

Thus, by induction every $K_{a,a} \setminus (6a - 34)$ graph is IK for all $a \geq 9$. \square

Theorem 17. *Any graph of the form $K_{a+n,a} \setminus (3n + 6a - 34)$, where $a \geq 9$ and $n \geq 0$, is IK.*

Proof: We use induction on n .

For the base case, let $n = 0$. This gives us a $K_{a,a} \setminus (6a - 34)$ graph, which is IK by Theorem 16.

Now, for the inductive step, assume that $K_{a+n,a} \setminus (3n + 6a - 34)$ is IK. Let's look at $K_{a+n+1,a} \setminus (3(n+1) + 6a - 34)$, or equivalently, $K_{a+n+1,a} \setminus (3n + 6a - 31)$. Using Lemma 6, $K_{a+n+1,a} \setminus (3n + 6a - 31)$ is IK provided $3n + 6a - 31 > 2(a + n + 1)$. This holds when $a = 9$ since $3n + 23 > 2(n + 10)$. Each time a increases by 1, $3n + 6a - 31$ increases by 6 while $2(a + n + 1)$ increases by 2. By induction, the inequality in Lemma 6 holds for all a and therefore each $K_{a+n+1,a} \setminus (3n + 6a - 31)$ graph is IK. \square

5. PROOF OF THEOREM 4

In this section we prove

Theorem 4. *Let a_n be defined by the recurrence*

$$a_n = \left\lfloor \frac{n(a_{n-1} - 1)}{n - 5} \right\rfloor + 1$$

when $n \geq 7$, $a_5 = 5$, and $a_6 = 7$. Let $C_n = a_n - 4n$, for $n \geq 7$, and $C_5 = C_6 = -17$. Let G be a bipartite graph with exactly $n \geq 5$ vertices in one part and at least a_n vertices in the other. If $e(G) \geq 4v(G) + C_n$ then G is IK.

Proof: Since $C_5 = C_6 = -17$ was proved in Theorem 2, we will assume that $n \geq 7$. It is enough to show that

Claim 1. *All $K_{n,a} \setminus ((n-4)a - a_n)$ graphs with $a \geq a_n$ are IK.*

Let us first observe that this will complete the argument. Indeed, let G be a $K_{n,a} \setminus m$ graph where $a \geq a_n$ and suppose $e(G) \geq 4v(G) + C_n$. That is, $na - m \geq 4(n + a) + C_n$. Then,

$$\begin{aligned} m &\leq na - 4(n + a) - C_n \\ &= na - 4(n + a) - (a_n - 4n) \\ &= (n - 4)a - a_n. \end{aligned}$$

So, if we can establish our claim, it will follow that G is IK, as required.

We prove the claim by induction. The case where $n = 6$ corresponds to Theorem 11. So, let $n \geq 7$.

We first observe that the claim is valid when $a = a_n$. Indeed, by the inductive hypothesis, all $K_{n-1,a_n} \setminus (n - 1 - 4)a_n - a_{n-1}$ graphs are IK. Then, applying Lemma 6, it follows that all $K_{n,a_n} \setminus (n - 5)a_n$ graphs are IK, as required.

Now use Lemma 6 to show, by induction, that the claim holds for all $a \geq a_n$. \square

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DEPARTMENT OF MATHEMATICS AND STATISTICS, CALIFORNIA STATE UNIVERSITY, CHICO, CHICO, CA 95929-0525

E-mail address: sophy26@hotmail.com

E-mail address: tinyapple@gmail.com

E-mail address: TMattman@CSUChico.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, 3620 SOUTH VERMONT AVE., KAP 108 LOS ANGELES, CA 90089-2532

E-mail address: `mmanriqu@usc.edu`